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# SUSY Gauge Dynamics and Singularities of 4d N=1 String Vacua

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Many N=1 heterotic string compactifications exhibit physically mysterious singularities at codimension one in the moduli space of vacua. At these singularities, Yukawa couplings of charged fields develop poles as a function of the moduli. We explain these conformal field theory singularities, in a class of examples, as arising from non-perturbative gauge dynamics of non-perturbative gauge bosons (whose gauge coupling is the sigma model coupling) in the string theory.

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## 1. Introduction

Recent progress in string duality raises hope for a quantitative nonperturbative understanding of generic string vacua. Much progress has been made in the special classes of models with extended supersymmetry (in the  $4d$  sense). In particular, previously mysterious singularities in conformal field theory moduli space (such as conifold singularities arising in type II theories compactified on Calabi-Yau manifolds [1] and small instanton singularities in  $SO(32)$  heterotic and type I theories [2]) have been explained as the result of additional (sometimes solitonic) states becoming massless at the singularity. For models with  $4d$   $N=1$  supersymmetry much less is known, as the supersymmetry is much less constraining. Some examples of dual pairs exist [3,4], but these examples are rather special as they arise as orbifolds of dual pairs with extended supersymmetry. It is the purpose of the present paper to begin the study of more generic  $N = 1$  string compactifications, by addressing the issue of singularities in the moduli space for a class of these models.

A heterotic  $N = 1$  model is obtained by compactification on an internal  $(0,2)$  superconformal field theory. Such models have been fruitfully studied using a gauged linear sigma model on the worldsheet [5,6], which flows in the infrared to a  $(0,2)$  superconformal field theory [7]. Singularities in the conformal field theory arise at codimension one loci in the linear sigma model moduli space where some multiplet(s) become free on the worldsheet, leading to a divergence in the path integral [5]. This leads in particular to a simple pole in a Yukawa coupling  $\kappa$  [7]. For the special case of  $(2,2)$  theories, this is the familiar pole which occurs at the conifold [8]. If  $z$  is a coordinate on the moduli space such that  $z \rightarrow 0$  is the singularity, then we find

$$\kappa \sim \frac{g^3}{z} f(\text{Moduli}) \quad (1.1)$$

where  $g^3$  is the invariant coupling of the relevant charged fields and  $f$  is some holomorphic function of the moduli, nonsingular in the limit.

Because the conformal field theory has become singular, it is natural to suspect that extra states are becoming light. Their dynamics should then explain the singular behavior of the conformal theory. The singularity (1.1) of the classical  $N = 1$  theory cannot arise from a perturbative effect, as occurred in the resolution of the analogous logarithmic singularity at the  $N = 2$  conifold [1]. Instead, the quantum resolution here must involve nonperturbative dynamics of the new light particles.

Poles in the superpotential do arise dynamically in supersymmetric QCD with the number of flavors one less than the number of colors [9]. In this paper we will discover a detailed relation between these two sets of poles. In section 2 we explain how a simple asymptotically free enhancement to the spectrum,  $SU(2)$  with four doublet chiral multiplets, can reproduce the singularity (1.1), with a particular function  $f$ . In section 3 we discuss in general terms a class of  $N=1$  heterotic string models—compactifications of the  $SO(32)$  heterotic string theory on Calabi-Yau threefolds which are  $K3$  fibrations—with codimension one “conifold” singularities. We derive the spectrum of light particles at the singularity and find it agrees with that discussed in §2!

The derivation of the spectrum in such a model proceeds as follows. On the  $(0,2)$  moduli space, there are codimension one singularities which consist of a small instanton on the  $K3$  fiber fibered over the  $\mathbf{P}^1$  base  $C$ . Because the nonperturbative enhancement of the spectrum of the  $6d$  theory compactified on the generic fiber is known [2], we can deduce the spectrum for the  $4d$   $N = 1$  theory on the fibration using similar methods to those employed in the  $N = 2$  case in [10,11]. That is, we compute the massless spectrum in four dimensions by determining the zero modes of the relevant Dirac operator on  $C$ .

The result one obtains for this class of models is a nonperturbative enhancement of the spectrum consisting of  $SU(2)$  gauge symmetry with four doublets.<sup>1</sup> Instanton effects in the  $SU(2)$  gauge theory explain the singularities of the conformal field theory. Note that the singularity occurs at string tree level in the heterotic theory. This can be reproduced by the dynamics of the  $SU(2)$  gauge bosons since they are non-perturbative, and their interactions are governed by the sigma model coupling. When the type I theory is compactified on the same space<sup>2</sup>, the extra states arise perturbatively in the DD and DN sector of open strings in a fivebrane background, and the dynamically induced superpotential is nonperturbative in  $g_{string}$ . The situation is very analogous to the one encountered in 4d  $N=2$  duality where the type II sigma model sums up nonperturbative effects for the heterotic string vector multiplets [13]. Here, the heterotic string sigma model computes nonperturbative corrections which can be ascribed to perturbative type I states. In section 4 we present in some detail an explicit example.

<sup>1</sup> This result applies on the generic  $(0,2)$  locus, where the six-dimensional singularity is a small instanton. It would be interesting to determine the behavior on the  $(2,2)$  locus, where for example the tree-level metric takes the  $N = 2$  form.

<sup>2</sup> In a compactification to six dimensions on  $K3$  this theory is dual to the corresponding heterotic theory. However, in four dimensions this might not be the case. The generic theory of this kind has “anomalous  $U(1)$ ” factors and develops Fayet-Iliopoulos D-terms [12] which lead to a shift in the vacuum. It is not clear to us whether heterotic-type I duality is true in this case.

## 2. SUSY QCD and the Singularity

Supersymmetric QCD with  $N_F = N_C = 2$  has a smooth quantum moduli space of vacua which is a deformation of the classical one. If we denote the four doublets  $d_i$ ,  $i = 1, \dots, 4$ , we can form gauge-invariant coordinates  $V_{ij} = d_i^\alpha \epsilon_{\alpha\beta} d_j^\beta$  where  $\alpha, \beta$  are  $SU(2)$  indices. Then the classical moduli space is given by  $Pf(V) = \epsilon^{ijkl} V_{ij} V_{kl} = 0$ . Quantum mechanically, a one instanton effect removes the singularity:

$$Pf(V) = \Lambda^4 \quad (2.1)$$

where  $\Lambda$  is the dynamical scale of the gauge theory [14]. The moduli space is five-complex-dimensional (with the  $SU(2)$  gauge symmetry Higgsed at generic points). The constraint (2.1) can be enforced with a Lagrange multiplier term in the superpotential:

$$W_0 = \lambda(Pf(V) - \Lambda^4) \quad (2.2)$$

In our problem there is one relevant dimension in the moduli space, the coordinate  $z$  (1.1). For this to agree with the gauge theory we need to add certain tree-level non-renormalizable interactions to the superpotential (2.2). In particular, consider adding the terms  $V_{13}^2 + V_{23}^2 + V_{14}^2 + V_{24}^2$  (the particular form of these terms is not crucial for our analysis). Integrating out  $V_{13}, V_{23}, V_{14}$ , and  $V_{24}$  we find that they all vanish. Then the Pfaffian constraint obtained by integrating out  $\lambda$  requires  $V_{34} = \frac{\Lambda^4}{V_{12}}$ , yielding a one-dimensional moduli space.

Now what about the pole? Consider adding the term  $g^3 V_{34}$  to the superpotential so that altogether we have

$$\begin{aligned} W = & g^3 V_{34} + \lambda(V_{12} V_{34} - V_{13} V_{24} + V_{14} V_{23} - \Lambda^4) \\ & + V_{13}^2 + V_{23}^2 + V_{14}^2 + V_{24}^2 \end{aligned} \quad (2.3)$$

Integrating out  $V_{13}, V_{23}, V_{14}$ , and  $V_{24}$  we still find that they all vanish. Furthermore, the constraint  $V_{34} = \frac{\Lambda^4}{V_{12}}$  yields upon substitution

$$W_{dyn} = \frac{g^3 \Lambda^4}{V_{12}} \quad (2.4)$$

This is the desired pole, where the function  $f$  from (1.1) is given by  $\Lambda^4$  as a function of moduli. This will be elucidated in the following sections, where we will see this phenomenon realized in  $K3$  fibration models.<sup>3</sup>

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<sup>3</sup> This is assuming that tree-level nonrenormalizable interactions of the sort in (2.3) arise given the spectrum; we have not performed the necessary computations to check this directly, but such terms must arise in order to reproduce the correct dimension of the moduli space.

Because it occurs for  $V_{34} \rightarrow \infty$ , in the gauge theory this pole appears to be at infinite distance in the moduli space. This would be in contradiction with known results for (2,2) models [15]. In our problem this gauge theory occurs as a nonperturbative enhancement at the singularity in the conformal field theory. The validity of the gauge theory analysis is limited to vacuum expectation values  $V_{ij} \ll M_S^2$  where  $M_S$  is the string scale. Therefore, the distance to the pole is not determined by this analysis. To be precise, this limitation is determined as follows. At large  $V_{34}$ , the Kahler potential for the theory (2.2) is given by its classical value (since all the fields are Higgsed):  $K \sim |V_{34}|$ . The gauge theory breaks down for  $V_{34} \sim M_S^2$  or equivalently  $V_{12} \sim \frac{\Lambda^4}{M_S^2}$ . The distance from this point to a finite point  $A$  in the moduli space is

$$d \sim \int_{\frac{\Lambda^4}{M_S^2}}^A \frac{d|V_{12}|}{|V_{12}|^{\frac{3}{2}}} \Lambda^2 = -\frac{\Lambda^2}{\sqrt{A}} + M_S \quad (2.5)$$

which is of course finite. The distance to the pole is not calculable in the field theory approximation.

### 3. Singularities in K3 Fibrations and SUSY QCD

In this section we will discuss in general terms the string models for which the desired spectrum at the singularity,  $SU(2)$  with four doublets, can be derived. In the next section we will provide the details for an example. Consider the  $SO(32)$  heterotic or type I string compactified on a K3 fibration, with holomorphic vector bundle  $V$ . In the adiabatic limit of large  $\mathbf{P}^1$  base size, such a model appears locally like a compactification to six dimensions on the fiber theory. Singularities can occur when the vector bundle or manifold becomes singular. In general one might expect singularities at codimension one to occur at isolated points in the manifold.

In fact, for K3 fibrations there is a codimension one component of the singular locus for which the theory on the generic fiber is singular. On the (2,2) locus, for example, one finds at codimension one an  $A_1$  singularity fibered over the base, as studied by [10,11] in compactifications of type II string theory. In the (0,2) context, singularities generically involve the vector bundle but not the manifold becoming singular. The codimension one singularity of the generic fiber in a (0,2) model will consist of a single small instanton.

It was demonstrated in [2] that the six-dimensional theory obtained by compactifying the heterotic or type I string on K3 with such a singularity in the vector bundle has an

enhanced symmetry: One finds  $SU(2)$  with hypermultiplets in the  $(\mathbf{32}, \mathbf{2})$  of  $SO(32) \times SU(2)$ . In the type I theory the small instanton is a D-5-brane; the  $SU(2)$  gauge bosons come from the DD (or 55) sector while the hypermultiplets arise in the DN (or 59) sector. On the heterotic side, the enhanced  $SU(2)$  is non-perturbative instead of arising in the perturbative string spectrum. In the next subsections we will show that the  $SU(2)$  gauge symmetry along with four doublets survives reduction to the 4d  $N=1$  theory obtained by fibration over a sphere  $C$ . We will focus on the heterotic string, and will denote this nonperturbative enhancement of the gauge field spectrum  $SU(2)_{NP}$ .

Before embarking on the derivation of this spectrum, let us discuss its relation to the gauge dynamics explained in section 2. In six dimensions, the  $SU(2)_{NP}$  gauge fields have a kinetic term which (in the string frame) is independent of the string coupling [16]. Upon reduction on  $C$  with radius  $R$ , this leads to an effective four-dimensional  $SU(2)$  gauge coupling  $\frac{1}{g_{NP}^2} \propto \frac{R^2}{\alpha'}$ . Then

$$\Lambda \propto M_S e^{-\frac{R^2}{\alpha'}} \quad (3.1)$$

Since the relevant part of the moduli space is one dimensional (the single modulus given by  $z$  in (1.1)), the four doublets must have interactions giving a tree-level superpotential such as (2.3) which ensures that. It would be interesting (though difficult) to compute these interactions in the type I theory using conformal field theory. Assuming such terms are there, we obtain the result (2.4). In particular, the function  $f$  in (1.1), which was determined in (2.4) to be  $\Lambda^4$ , depends only on the moduli and not on the string coupling, as required for a conformal field theory effect. So we see that the spacetime instanton effect (2.4) computes for us nonperturbative effects on the worldsheet of the heterotic string.

Putting together the gauge theory and string theory, we have the following hierarchy of scales. Above the scale  $1/R$  the theory is six-dimensional and was studied in [2]. Below the scale  $1/R$ , but above the scale  $\Lambda$ , the model is four-dimensional and lives on the classical moduli space of  $SU(2)$  with four doublets. Including the instanton effect leads to the deformation (2.2) of the moduli space, removing the singularity at the origin. From the worldsheet point of view this reflects the fact that worldsheet instantons, which wrap around the entire base, probe beyond the local adiabatic regime and can alter the structure of the moduli space inherited from six dimensions. Note that in the present context of  $K3$  fibrations, the coupling (1.1) goes to zero in the limit of  $\Lambda \rightarrow 0$ . The six-dimensional theory has no such coupling, so this form of  $f$  is consistent.

Let us now proceed with the computation of the spectrum. As noted in the type II context by [10,11], the nontrivial fibration of the K3 theory over the base C is crucial in obtaining a consistent supersymmetric theory in 4d, as C alone is not flat. For  $N = 1$  supersymmetry we require one covariantly constant spinor to survive the reduction. This is possible due to the twisting of the Dirac operator on C arising from the Lorentz and gauge transformation properties of the fields in the full Calabi-Yau. The twisting to obtain  $N = 1$  supersymmetry occurs as follows. The Lorentz group  $SO(4)$  in the internal four dimensions of the K3 decomposes as  $SO(4) = SU(2)_H \times SU(2)'$  where  $SU(2)_H$  is the holonomy group of the K3. The components of the six-dimensional supercharges have charges  $\pm 1/2$  under the Lorentz  $U(1)$  on C and transform as a **2** of  $SU(2)'$ . Therefore a twist by the  $U(1)$  generator  $J'_3$  in  $SU(2)'$  preserves half of the 6d  $N = 1$  supersymmetry, giving  $N = 1$  supersymmetry in four dimensions. Meanwhile the  $N = 1$   $SU(2)$  vector multiplets are singlets under Lorentz and  $SO(32)$  gauge symmetries, as well as under the Lorentz rotations on C. Thus they survive on reduction as constants on C.

### 3.1. The Charged Matter Spectrum from the Splitting Type of V

More analysis is required to find the components of the 6d hypermultiplets which survive as chiral multiplets in four dimensions. In the type I theory, the nonperturbative spectrum lives on a D-brane which is partially wrapped around the curve C. The origin of the hypermultiplets in quantizing the D-brane is explained in section 3.1 of [2]. Let  $2, \dots, 5$  be the coordinates on the 5-brane in light cone gauge and  $6, \dots, 9$  the normal coordinates. In the DN Neveu-Schwarz sector (which gives spacetime bosons) the vacuum is a spinor of  $SO(4)_{6, \dots, 9}$  and a scalar of  $SO(4)_{2, \dots, 5}$ . In the DN Ramond sector (which gives spacetime fermions) the vacuum is a scalar of  $SO(4)_{6, \dots, 9}$  and a spinor of  $SO(4)_{2, \dots, 5}$ .

Let us count the fermions; the bosons are then given by the surviving  $N = 1$  supersymmetry just discussed. We have 32 hypermultiplets, each of which has two chiral multiplets. Six of the 32 hypermultiplets lie in the vector bundle  $V, V^*$  of the CY theory. Let us first consider these: We need zero modes of the Dirac operator acting on these fermions, which are given by  $H^0(\mathcal{O}(-1) \otimes V)$  and  $H^0(\mathcal{O}(-1) \otimes V^*)$ . Any holomorphic vector bundle on  $\mathbf{P}^1$  splits into a sum of holomorphic line bundles [17]. These line bundles are denoted  $\mathcal{O}(k)$ , where  $k$  is the integrated first Chern class  $\int_{\mathbf{P}^1} c_1(\mathcal{O}(k)) = k$ . For  $k \geq 0$ ,  $\mathcal{O}(k)$  has  $k + 1$  sections; for  $k < 0$  there are none. In general, for a rank three bundle,

$$V \Big|_C = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \quad (3.2)$$

and

$$V^* \Big|_C = \mathcal{O}(-a) \oplus \mathcal{O}(-b) \oplus \mathcal{O}(-c). \quad (3.3)$$

where  $c_1(V) \propto a + b + c \equiv 0$ . We will show in the next section that for (a class of) K3 fibration models,  $a = 2$ ,  $b = 0$ , and  $c = -2$ . Tensoring with  $\mathcal{O}(-1)$ , we have  $N_\psi$  zero modes, where

$$N_\psi = 2 \times \left( h^0[\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-3)] + h^0[\mathcal{O}(-3) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)] \right) \quad (3.4)$$

so

$$N_\psi = 2 \times ([2 + 0 + 0] + [0 + 0 + 2]) = 8 \quad (3.5)$$

Since 8 chiral fermions makes for 8 chiral multiplets, we will have 4  $SU(2)$  doublets in the surviving spectrum. Note that without the above twist of the spin bundle  $\mathcal{O}(-1)$  by the vector bundle  $V$  with nontrivial splitting type there would be no sections, and none of the hypermultiplets would survive. In particular, the other 26 hypermultiplets do not survive, so we are left with precisely 4 doublets.

#### 4. Explicit Example

The best-studied class of  $N = 1$  heterotic string vacua consists of those models that can be realized as the infrared limit of a gauged linear sigma model on the worldsheet [5,6,7]. Many such examples have a phase which is geometrical, in that (for large values of a parameter) the model reduces to a nonlinear sigma model on a compactification manifold. This manifold  $K$  must be Calabi-Yau and equipped with a stable holomorphic vector bundle  $V$  in order to satisfy the large-radius conditions for conformal invariance. Conformal invariance throughout the linear sigma model moduli space was demonstrated in [18,7]. The structure of singularities in such (0,2) models is manifest in the linear sigma model description [5]. In this description the moduli appear as coupling constants in the two-dimensional sigma model.

Motivated by the considerations of the previous sections, let us consider an example which admits a relation to six dimensions, namely a K3 fibration. One example is a (0,2) model on the hypersurface of degree 8 in  $\mathbf{WP}_{11222}^4$ .<sup>4</sup> Take the defining equation

$$G_0(z_1, \dots, z_5) = \frac{z_1^8}{8} + \frac{z_2^8}{8} + \frac{z_3^4}{4} + \frac{z_4^4}{4} + \frac{z_5^4}{4} + \dots = 0. \quad (4.1)$$

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<sup>4</sup> In fact, the following analysis (with obvious modifications) will hold for all K3 fibration models with weights of the form  $(1, 1, 2k_1, 2k_2, 2k_3)$ .

This hypersurface has a curve of  $A_1$  singularities inherited from the  $\mathbf{WP}_{11222}^4$ :

$$\left\{ z_i = (0, 0, z_3, z_4, z_5) \mid \frac{z_3^4}{4} + \frac{z_4^4}{4} + \frac{z_5^4}{4} + \dots = 0 \right\} \quad (4.2)$$

Resolving this singularity involves inserting a  $\mathbf{P}^1$  for each singular point on the curve. The precise procedure will become clear momentarily from the construction of the linear sigma model for this compactification. To be specific we will take  $V$  to be the rank 3 deformation of the tangent bundle on this hypersurface.

The  $(0,2)$  linear sigma model describing compactification on this hypersurface has the following multiplets and interactions. We will work in  $(0,2)$  superspace with fermionic coordinates  $\theta^+$  and  $\bar{\theta}^+$  and bosonic coordinates  $y^\alpha$ ,  $\alpha = 1, 2$ . The imaginary part of the Kahler parameter,  $r_1$ , arises as the coefficient of a Fayet-Iliopoulos  $D$ -term of a worldsheet  $U(1)$  gauge group with Fayet-Iliopoulos parameter  $r_1$ . There is an additional Kahler parameter  $r_2$  which determines the size of the  $\mathbf{P}^1$  resolving (4.2), leading to a second  $U(1)$  gauge multiplet on the worldsheet. We will call these two Abelian factors  $U(1)_A$ , where  $A = 1, 2$ .

The superpotential terms will involve seven chiral superfields. The first set corresponds to the coordinates  $z_i = (z_1, \dots, z_5)$ :

$$Z^i = z^i + \sqrt{2}\theta^+ \psi_+^i - i\theta^+ \bar{\theta}^+ (D_0 + D_1) z^i, \quad i = 1, \dots, 5 \quad (4.3)$$

These will have charges  $(1,1,2,2,2)$  under  $U(1)_1$  and  $(1,1,0,0,0)$  under  $U(1)_2$ . Because we are considering a  $(0,2)$  deformation of a  $(2,2)$  model, each chiral multiplet has an associated left-moving fermionic multiplet

$$\Lambda_-^i = \lambda_-^i - \sqrt{2}\theta^+ G^i - i\theta^+ \bar{\theta}^+ (D_0 + D_1) \lambda_-^i - 2i\bar{\theta}^+ Q_A^A \Sigma'_A Z^i \quad (4.4)$$

with the same gauge charges  $Q_A^i$  as the corresponding  $Z^i$ . Here  $G^i$  is an auxiliary field which gets integrated out in favor of the  $(0,2)$  superpotential term  $\bar{J}_i$  which is introduced below. (In  $(2,2)$  language, the  $Z^i$  and  $\Lambda_-^i$  would combine into ordinary chiral superfields.) The  $\Lambda_-^i$  obey a chirality condition

$$\bar{D}_+ \Lambda_-^i = E^i = 2iQ_A^i \Sigma'_A Z^i. \quad (4.5)$$

where  $E^i$  are holomorphic functions of the chiral superfields.

In addition we will have two more chiral multiplets (and associated left-moving fermionic multiplets)  $(p, \lambda_-^p, \psi_+^p)$  with charges (-8, 0) under the two U(1)'s and  $(x, \lambda_-^x, \psi_+^x)$  with charges (0,-2).

The Lagrangian consists of the standard flat kinetic terms together with the  $U(1)$   $D$ -terms and  $\theta$ -terms, as well as superpotential terms. The superpotential terms are

$$L_J = -\frac{1}{\sqrt{2}} \int d^2y \left. d\theta^+ \Lambda_-^I J_I \right|_{\bar{\theta}^+=0} + h.c. \quad (4.6)$$

where  $I$  indexes the chiral superfields  $Z_i, P, X$ . Here

$$J_p = G(Z^i, X) \quad (4.7)$$

where  $G(Z^i, X) = 0$  is the defining equation for the resolved hypersurface:

$$G(z_i, x) = \frac{z_1^8 x^4}{8} + \frac{z_2^8 x^4}{8} + \frac{z_3^4}{4} + \frac{z_4^4}{4} + \frac{z_5^4}{4} + \dots \quad (4.8)$$

The  $\dots$  refers to the other gauge-invariant terms that can occur in the  $\Lambda_-^p$  term of (4.6).

In the other terms,  $J_i \equiv p\tilde{J}_i$ ,  $J_x \equiv p\tilde{J}_x$ , where  $\tilde{J}_i$  and  $\tilde{J}_x$  are homogeneous polynomials in  $z_i, x$  of the appropriate degrees to render the terms gauge invariant. They satisfy the conditions

$$Q_1^i z_i J^i = 8PG(z^i) \quad (4.9)$$

and

$$z_1 J^1 + z_2 J^2 - 2x J_x = 0. \quad (4.10)$$

This ensures (using (4.5) and the values of the charges) that  $\sum_I E^I J_I = 0$ , so that (4.5) is consistent with (4.6) having (0,2) supersymmetry. The model actually has (2,2) supersymmetry if and only if  $J_i = P \frac{\partial G}{\partial Z^i}$  and  $J_x = P \frac{\partial G}{\partial x}$  since then the superpotential term (4.6) can be written in (2,2) superspace as  $\int d^2\theta PG$ . Departing from this locus breaks (2,2) supersymmetry to (0,2) and has the effect of perturbing the tangent bundle of the hypersurface to a more general bundle  $V$ . We will work explicitly with such a (0,2) deformation below. (Since the perturbed bundle has rank three, the space-time gauge group is still  $SO(26) \times U(1)$ .) If we decompose  $\tilde{J}_i$  as  $\tilde{J}_i = \frac{\partial G}{\partial s^i} + G_i$  for some homogeneous polynomials  $G_i$  with the appropriate gauge charges satisfying  $G_i z^i = 0$ , then the parameters in  $G_i$  are the moduli that break (2,2) down to (0,2).

The bosonic potential for this model is

$$\begin{aligned}
U(\phi_I) &= \frac{e^2}{2} \sum_A \left( \sum_I Q_A^I |\phi_I|^2 - r_A \right)^2 + \sum_{A,B=1}^2 \sum_I Q_A^I Q_B^I |\phi_I|^2 \sigma_A \bar{\sigma}_B + \sum_I |J_I|^2 \\
&= \frac{e_1^2}{2} \left( |z_1|^2 + |z_2|^2 + 2|z_3|^2 + 2|z_4|^2 + 2|z_5|^2 - 8|p|^2 - r_1 \right)^2 \\
&\quad + \frac{e_2^2}{2} \left( |z_1|^2 + |z_2|^2 - 2|x|^2 - r_2 \right)^2 \\
&\quad + |\sigma_1 + \sigma_2|^2 \left( |z_1|^2 + |z_2|^2 \right) + |\sigma_1|^2 \left( 64|p|^2 + 4|z_3|^2 + 4|z_4|^2 + 4|z_5|^2 \right) \\
&\quad + 4|\sigma_2|^2 |x|^2 + |G|^2 + \sum_i |J_i|^2 + |J_x|^2
\end{aligned} \tag{4.11}$$

The first two terms on the right hand side come from integrating out the auxiliary fields  $D_A$ .

The theory can be studied semiclassically at large  $|r_A|$ ; for  $r_A \gg 0$  we find the Calabi-Yau phase in which the linear sigma model describes string propagation on the hypersurface of degree 8 (4.8) in  $\mathbf{WP}_{11222}^4$ . For generic values of  $r_A$  and the parameters defining the polynomials  $J_I$ , the model is nonsingular: there is a nonvanishing potential as the fields become large in any direction in field space.<sup>5</sup>

At complex codimension one in the linear sigma model moduli space one finds a singular locus for which a direction in field space exists with vanishing potential for large field strength. For example, when  $t_2 \rightarrow 0$ ,  $\sigma_2$  can become arbitrarily large with no cost in potential as long as  $z_1 = z_2 = x = 0$ . It is another singularity which will be of interest to us: the one which appears where  $t_1 = t_2$ . There  $\sigma_1$  and  $\sigma_2$  can become large with no cost in potential as long as  $\sigma_1 + \sigma_2 = 0 = z_3 = z_4 = z_5 = p = x$ .

One quantity of physical interest in the low-energy effective  $N = 1$  supergravity theory is the set of Yukawa couplings of the generations and antigenerations charged under the unbroken part of the  $SO(32)$  gauge group. In the present case these are the couplings  $\mathbf{26}_{\pm 1} \mathbf{1}_{\mp 2} \mathbf{26}_{\pm 1}$  where we have indicated the gauge charges under the unbroken  $SO(26) \times U(1)$ . In our example there are 2 generations (related on the (2,2) locus by left-moving

<sup>5</sup> As discussed in [7], it is not true that the potential *grows* in every direction in field space, as setting  $\phi_I = 0$  leaves a constant action  $\frac{(\text{Area of worldsheet}) \times e^2}{2} \sum_A (r_A^2 + (\frac{\theta_A}{2\pi})^2)$  [7]. This does not lead to a divergence in the path integral for the physical limit in which the area of the worldsheet goes to infinity.

supersymmetries to the Kahler moduli  $r_1$  and  $r_2$ ) and 86 antigenerations. As discussed in [7], the linear sigma model vertex operators for the generations are (linear combinations of)  $\sigma_1$  and  $\sigma_2$ . The simple Hamiltonian computation of [7] reveals that the cubic coupling of  $\sigma_1 - \sigma_2$  has a simple pole in  $t_1 - t_2$  as  $t_1 - t_2 \rightarrow 0$  as in (1.1). For (0,2) models, singularities which occur in the large radius phase generically involve the vector bundle  $V$  degenerating while the manifold remains smooth. Singularities such as the present one which occur in the (worldsheet) quantum regime in (0,2) models are not accessible by a (0,2) generalization of mirror symmetry but can be studied fruitfully in the linear sigma model as just indicated. On the (2,2) locus, one finds a logarithmic singularity in the metric on the moduli space; the behavior of the metric is not known for (0,2) models.

#### 4.1. The Theory on the Generic Fiber

The manifold (4.8) is a K3 fibration, as can be seen from the field redefinition

$$z_1 = \lambda z_2, \quad \lambda_-^1 = \lambda \lambda_-^2, \quad y \equiv x z_2^2 \quad (4.12)$$

upon which the defining equation becomes

$$G_f(y, z_3, z_4, z_5) = \frac{(1 + \lambda^8)y^4}{8} + \frac{z_3^4}{4} + \frac{z_4^4}{4} + \frac{z_5^4}{4} + \dots = 0. \quad (4.13)$$

This is the defining equation for the quartic hypersurface in  $\mathbf{CP}^3$ , one algebraic realization of K3.

In the heterotic context the fiber theory consists of the pair  $(K3, V_f)$ , where  $V_f$  is the vector bundle to which the left movers in the fiber theory couple. Therefore in addition to deducing the fiber manifold (4.13), we must also extract the fiber theory's vector bundle polynomials (i.e. the K3 theory's version of the  $J_I$  in the Calabi-Yau theory), which we will denote  $F_1, \dots, F_4$ . Let us work this out starting from a general vector bundle in the Calabi-Yau theory:

$$J_1 = p[z_1^7 x^4 + G_1] \quad (4.14)$$

$$J_2 = p[z_2^7 x^4 + G_2] \quad (4.15)$$

$$J_3 = p[z_3^3 + G_3] \quad (4.16)$$

$$J_4 = p[z_4^3 + G_4] \quad (4.17)$$

$$J_5 = p[z_5^3 + G_5] \quad (4.18)$$

$$J_x = \frac{p}{2}[(z_1^8 + z_2^8)x^3 + G_x] \quad (4.19)$$

We require here that (4.9)(4.10) be satisfied. This requires in particular that

$$2xJ_x = z_1 J_1 + z_2 J_2. \quad (4.20)$$

Using this, (4.12), and (4.6), the worldsheet superpotential terms involving  $J_1$ ,  $J_2$ , and  $J_x$  are

$$\mathcal{L}_{sup} = \int d\theta \left( (\Lambda_-^2 + \frac{\Lambda_-^x z_2}{2x})(\lambda J_1 + J_2) \right) + \dots \quad (4.21)$$

If we define

$$\chi_- \equiv \left( \Lambda_-^2 + \frac{\Lambda_-^x z_2}{2x} \right) z_2 x \quad (4.22)$$

and

$$F_1 \equiv \frac{\lambda J_1 + J_2}{z_2 x} \Big|_{z_1 = \lambda z_2, y = x z_2^2} \quad (4.23)$$

$$F_2 \equiv J_3(y, z_3, z_4, z_5) \quad (4.24)$$

$$F_3 \equiv J_4(y, z_3, z_4, z_5) \quad (4.25)$$

$$F_4 \equiv J_5(y, z_3, z_4, z_5) \quad (4.26)$$

then the superpotential becomes

$$\mathcal{L}_{sup} = \int d\theta \left( \Lambda_-^p G_f + \chi_- F_1 + \Lambda_3 F_2 + \Lambda_4 F_3 + \Lambda_5 F_4 \right). \quad (4.27)$$

Note that the definition (4.23) makes sense since for gauge invariance each term in  $G_1$  and  $G_2$  must contain at least one factor of  $z_1$  or  $z_2$  and at least one factor of  $x$ . In particular, (4.23)-(4.26) reduce to the tangent bundle of the K3 fiber when  $G_i = 0$  in the Calabi-Yau theory (4.14)-(4.19).

This superpotential determines the manifold and rank 2 bundle for the geometrical phase of large  $r_1$ . (Here  $r_2$  is of course taken to be large, as we are considering an adiabatic limit in which we can study the theory on the generic fiber over the  $\mathbf{P}^1$  whose size is given by  $r_2$  (4.11).) For  $t_1 - t_2$  small, which will be near the singularity of interest, this linear sigma model is strongly coupled and does not directly give a description in terms of classical geometry. Because we are on K3, which generically has no worldsheet instantons, this is only an artifact of the description, and in the infrared the model must reduce to some conformal field theory on K3 with a rank 2 vector bundle.

We would like to set up the model so that the theory on the generic fiber has a small instanton singularity when  $t_1 - t_2 \rightarrow 0$ , so that we can use the results of [2]. This will involve turning on the (0,2) moduli in the  $J_I$ , and hence (0,4) moduli in the fiber theory, since we are aiming for a singularity of the vector bundle on the fiber which is not accompanied by a singularity of the K3 manifold itself.

First consider the (4,4) locus, which we can study explicitly using mirror symmetry to map us to a model in which the classical geometry is evident, since K3 is self-mirror. One example of a description which we obtain in this way is the following (using the procedure discussed in [8] for the quintic threefold in  $\mathbf{CP}^4$ , translated one dimension down to the quartic twofold in  $\mathbf{CP}^3$ ). To obtain a mirror description we mod out by all phase symmetries preserving the holomorphic 2-form [19]. Set  $(y, z_3, z_4, z_5) \equiv (\eta(\lambda)x_1, \dots, x_4)$ , where  $\eta(\lambda) = (8/(1 + \lambda^8))^{\frac{1}{4}}$ . A basis for these phase symmetries is  $g_1 : (x_1, \dots, x_4) \rightarrow (x_1, \alpha x_2, x_3, \alpha^3 x_4)$  and  $g_2 : (x_1, \dots, x_4) \rightarrow (x_1, x_2, \alpha x_3, \alpha^3 x_4)$  where  $\alpha = \exp(\frac{2\pi i}{4})$ . This restricts us to one deformation theoretic modulus  $\rho - 1$ , which corresponds to the parameter  $t_1 - t_2$  in the original linear sigma model description:

$$G_{f,orb} = \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^4}{4} + \frac{x_4^4}{4} - \rho x_1 x_2 x_3 x_4 = 0. \quad (4.28)$$

We now make the following field redefinition

$$(x_1, \dots, x_4) \equiv (y_1 y_3^{\frac{1}{4}}, y_2^{\frac{3}{4}} y_4^{\frac{1}{4}}, y_3^{\frac{3}{4}} y_2^{\frac{1}{4}}, y_4^{\frac{3}{4}}) \quad (4.29)$$

which respects the phase symmetries. Then in terms of  $(y_1, \dots, y_4)$ , the defining equation becomes

$$\tilde{G}_f(y_1, \dots, y_4) = \frac{y_1^4 y_3}{4} + \frac{y_2^3 y_4}{4} + \frac{y_3^3 y_2}{4} + \frac{y_4^3}{4} - \rho y_1 y_2 y_3 y_4 = 0. \quad (4.30)$$

This describes the same K3 as a hypersurface of degree 27 in  $\mathbf{WP}_{5,6,7,9}^3$ .

Consider the following 2-parameter family of vector bundles:

$$\tilde{F}_1 = y_1^3 y_3 - \rho y_2 y_3 y_4 \quad (4.31)$$

$$\tilde{F}_2 = \frac{3}{4} y_2^2 y_4 + \frac{y_3^3}{4} - \rho y_1 y_3 y_4 + \frac{\delta}{4} y_3^3 \quad (4.32)$$

$$\tilde{F}_3 = \frac{3}{4} y_3^2 y_2 + \frac{y_1^4}{4} - \rho y_1 y_2 y_4 - \frac{\delta}{4} y_2 y_3^2 \quad (4.33)$$

$$\tilde{F}_4 = \frac{y_2^3}{4} + \frac{3}{4} y_4^2 - \rho y_1 y_2 y_3 \quad (4.34)$$

Here  $\delta$  is a  $(0,4)$  deformation which preserves the fact that the vector bundle satisfies the conditions analogous to (4.9)(4.10) in this case. The singularity on the  $(4,4)$  locus occurs at  $(\rho, \delta) = (1, 0)$ . Starting from this locus, turning on  $\rho - 1$  alone removes the singularity. A little algebra shows that turning on  $\delta$  alone also removes the singularity. Since (as is clear from the original linear sigma model for the fiber) the singularity is at codimension one in the linear sigma model moduli space, this means that some combination of  $\rho - 1$  and  $\delta$  preserves the singularity. Since turning  $\rho - 1$  on removes the singularity from the manifold, the resulting fiber theory has a singularity in the vector bundle  $\tilde{V}_f$  and not the manifold (4.30).

It is easy to check that each of the 45 hypermultiplet moduli of the  $SU(2)$  gauge bundle on  $K3$  has a complex deformation-theoretic representative in the original linear sigma model. This means in particular that the deformation  $\delta$  is accessible in the linear sigma model. Thus by turning on  $(0,2)$  moduli in the Calabi-Yau theory before taking the limit  $t_1 - t_2 \rightarrow 0$ , we can obtain a singularity which on the generic fiber is a singularity of the bundle but not the manifold. Singularities of the gauge bundle on  $K3$  occur when one or more instantons shrink to zero size. Since the singularity is codimension one in the linear sigma model, we expect there to be a single small instanton in the generic fiber for  $t_1 - t_2 \rightarrow 0$ .<sup>6</sup>

#### 4.2. Computation of the Splitting Type of $V$

As discussed in the last section, we need to compute the splitting type of  $V$  at the singularity in order to obtain  $H^0(\mathcal{O}(-1) \otimes V)$  and  $H^0(\mathcal{O}(-1) \otimes V^*)$  there. Because the left-moving fermions transform as sections of the spinor bundle on the worldsheet and as sections of  $V$  in spacetime, this is the same as determining the zero modes of the left-moving fermions of the linear sigma model in the background of one worldsheet instanton on the curve. In particular, we can compute the precise splitting type of  $V$  by the methods used in [20] for the quintic tangent bundle. The  $\mathbf{P}^1$  of interest is

$$(z_1, \dots, z_5; x) = (\lambda z_2, z_2, a, b, c; 0) \quad (4.35)$$

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<sup>6</sup> This is true unless the linear sigma model has “blown up” the singularity. But with a rank 2 bundle there are physical singularities at codimension one in the instanton moduli space. So if the codimension one singularity we see in the linear sigma model has been “blown up”, then there are others which have been “blown down” with respect to the physical moduli space. We will here assume that in the rank 2 case the linear sigma model gives an accurate description of the singularities.

Where  $a$ ,  $b$ , and  $c$  are constants satisfying  $\frac{a^4}{4} + \frac{b^4}{4} + \frac{c^4}{4} = 0$ . This satisfies (4.8) and gives the base of the fibration described in the last section. The structure of worldsheet instantons in the linear sigma model was analyzed in [5,21]. The instanton (4.35) has

$$\int v_{12}^{(2)} = 2\pi \quad ; \quad \int v_{12}^{(1)} - v_{12}^{(2)} = 0. \quad (4.36)$$

where  $v_{12}^{(A)}$  is the field strength for the worldsheet gauge group  $U(1)_A$ .

This means that the only fermions that can possibly have zero modes are those charged under  $U(1)_2$ , since the spinor bundle alone is  $\mathcal{O}(-1)$  and  $H^0(\mathcal{O}(-1)) = 0$ . In particular, only  $\lambda^1$ ,  $\lambda^2$ , and  $\lambda^x$  are charged under  $U(1)_2$ . Since  $J_1|_C = J_2|_C = 0$ ,<sup>7</sup>  $\lambda^1$  and  $\lambda^2$  reduce at low energies in the linear sigma model to a vector bundle transforming as does the tangent bundle of  $\mathbf{P}^1$  (tensored with the spinor bundle  $\mathcal{O}(-1)$ ), and therefore lie in  $\mathcal{O}(2) \otimes \mathcal{O}(-1)$  on  $C$ . Similarly, since  $\lambda^3$ ,  $\lambda^4$ ,  $\lambda^5$  transform as sections of  $\mathcal{O}(0) \times \mathcal{O}(-1)$  on  $C$ , and since  $J_3$ ,  $J_4$ , and  $J_5$  are constant on the curve, we can solve  $\lambda^3 J_3 + \lambda^4 J_4 + \lambda^5 J_5 = 0$  with  $\lambda^I \sim \lambda^I + Q_A^I \Phi^I \eta$ , and there is therefore a term  $\mathcal{O}(0)$  in  $V|_C$ . Since  $c_1(V) = 0$ , and  $V$  is rank 3, we immediately have that  $V|_C = \mathcal{O}(2) + \mathcal{O}(0) + \mathcal{O}(-2)$  as promised in (3.2). (This is the splitting type one would have for the tangent bundle, since there is a curve of  $\mathbf{P}^1$ 's in (4.2).) Thus we are left with  $SU(2)$  with four doublets as the nonperturbative enhancement of the spectrum for the  $N = 1$  theory at the “conifold”, as anticipated.

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<sup>7</sup> This holds even for a general (0,2) deformation because for gauge invariance, each term in  $J_1$  and  $J_2$  contains at least one power of  $x$ .

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